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The asymptotic property for nonlinear fourth-order Schrödinger equation with gain or loss

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Abstract

We study the Cauchy problem of the nonlinear fourth-order Schrödinger equation with gain or loss: $iu_t + \Delta^2 u + \lambda|u|^\alpha u + i\varepsilon a(t)|u|^\beta u = 0$, $x \in R^n$, $t \in R$, where $2 \le \alpha \le \frac{8}{n-4}$ and $2 \le \beta \le \frac{8}{n-4}$, ε is a real number, $a(t)$ is a real function, and $n > 4$. We study the asymptotic properties of its local and global solutions as $\varepsilon \to 0$.

Keywords: nonlinear fourth-order Schrödinger equation with gain or loss; Fourier restriction norm method; Cauchy problem

1 Introduction

In this paper we study the following nonlinear fourth-order Schrödinger equation with gain or loss:

$$\begin{cases} iu_t + \Delta^2 u + \lambda|u|^\alpha u + i\varepsilon a(t)|u|^\beta u = 0, & x \in R^n, t \in R, \\ u(x,0) = u_0(x), & x \in R^n, \end{cases} \tag{1.1}$$

where $u(x,t)$ are complex-valued function. We have $2 \le \alpha \le \frac{8}{n-4}$ and $2 \le \beta \le \frac{8}{n-4}$, ε is a real number, $a(t)$ is a real function, and $n > 4$.

For the case $\varepsilon = 0$, the above equation is the nonlinear fourth-order Schrödinger equation,

$$\begin{cases} iu_t + \Delta^2 u + \lambda|u|^\alpha u = 0, & x \in R^n, t \in R, \\ u(x,0) = u_0(x), & x \in R^n. \end{cases} \tag{1.2}$$

For (1.2), in [1] we have obtained the local well-posedness result in the space $C([-T,T], H^2(R^n))$ if $n > 4$ and $2 \le \alpha \le \frac{8}{n-4}$. We also get the global well-posedness result in the space $C(R, H^2(R^n))$ if $n > 4$ and $\lambda > 0$, $2 \le \alpha \le \frac{8}{n-4}$ or $\lambda < 0$, $2 \le \alpha \le \frac{8}{n}$. For the energy-critical case, in [2] and [3], Pausader Benoit gives the global well-posedness and scattering for $n \ge 5$ and radial initial data. In [4], Miao *et al.* study the defocusing case and obtain the global existence for $n \ge 9$. In [5], Zhang and Zheng obtain the global solution and scattering for $n = 8$. Pausader Benoit also discusses the mass-critical case in [6].

For the case $\varepsilon \neq 0$, $a(t)$ is the gain (loss) if $a(t) < 0$ ($a(t) > 0$). In [7], the authors discuss the Schrödinger equation with gain. They have obtained the result: The value of $a(t)$ will determine whether or not the solution will blow up. Feng *et al.* study the Schrödinger equation with gain/loss in [8] and [9]. They, respectively, give the limit behavior of solution as $\varepsilon \to 0$ and the global solution and blow-up result. As far as we know, there are fewer results about the fourth-order Schrödinger equation with gain. In this paper, we will discuss the local well-posedness and the global well-posedness of (1.1); especially, we will discuss the asymptotic behavior of the solution as $\varepsilon \to 0$.

2 The preliminary estimates

First, we denote by $U(t)$ $(t \in R)$ the fundamental solution operator of the fourth-order Schrödinger equation [10], *i.e.*,

$$U(t)\varphi(x) = F^{-1}\big(e^{-it\xi^4}\hat{\varphi}(\xi)\big) \quad \text{for } \varphi \in S'(R),$$

where $\hat{\varphi}$ denotes the Fourier transformation of φ, and F^{-1} represents the inverse Fourier transformation.

Thus the equivalent integral equations [11] of (1.1) and (1.2) are, respectively,

$$u_\varepsilon(t) = U(t)u_0 + i\lambda\int_0^t U(t-s)\big(|u_\varepsilon|^\alpha u_\varepsilon\big)(s)\,ds - \varepsilon\int_0^t U(t-s)a(s)\big(|u_\varepsilon|^\beta u_\varepsilon\big)(s)\,ds \tag{2.1}$$

and

$$u(t) = U(t)u_0 + i\lambda\int_0^t U(t-s)\big(|u|^\alpha u\big)(s)\,ds. \tag{2.2}$$

Second, we introduce the following notations. For any given $T > 0$, we define the space $L^q(0,T;W^{2,r}(R^n))$ with the norm

$$\|u\|_{L^q(0,T;W^{2,r})} := \left(\int_0^T \big\|u(\cdot,t)\big\|_{W^{2,r}(R^n)}^q\,dt\right)^{\frac{1}{q}}.$$

For two integers $8 \le q \le \infty$ and $2 \le r < \infty$, we say that (q,r) is an admissible pair if the following condition is satisfied:

$$\frac{2}{q} = \frac{n}{4}\left(1 - \frac{2}{r}\right).$$

For simplicity, in this paper, we will use C to denote various constants which may be different from line to line.

We have the following Strichartz estimate (see [1]): For any admissible pair (q,r)

$$\big\|U(t)\varphi(x)\big\|_{L^q(0,l;L^r)} \le C\|\varphi\|_{L^2} \tag{2.3}$$

and

$$\left\|\int_0^t U(t-s)f(x,t)\,ds\right\|_{L^q(0,l;L^r)} \le C\|f\|_{L^{\gamma'}(0,l;L^{\rho'})}, \tag{2.4}$$

where (γ,ρ) is an arbitrary admissible pair, and $'$ represents the conjugate number.

From Theorem 4.5 of [1], we have the following results.

Proposition 2.1 (subcritical case) *Assume that $n > 4$, $a \in L^\infty(0,\infty)$, $2 \le \alpha < \frac{8}{n-4}$, and $2 \le \beta < \frac{8}{n-4}$, $(\gamma_1,\rho_1) = (\alpha+2, \frac{2n(\alpha+2)}{n(\alpha+2)-8})$, $(\gamma_2,\rho_2) = (\frac{8(\beta+2)}{n\beta}, \beta+2)$. For any $u_0 \in H^2(R^n)$, there exists δ such that the Cauchy problem* (1.1) *has a unique solution u_ε in the space $L^\infty(0,\delta;H^2(R^n)) \cap L^{\gamma_1}(0,\delta;W^{2,\rho_1}(R^n)) \cap L^{\gamma_2}(0,\delta;W^{2,\rho_2}(R^n))$. Moreover,*

$$\|u_\varepsilon\|_{L^\infty(0,\delta;H^2)\cap L^{\gamma_1}(0,\delta;W^{2,\rho_1})\cap L^{\gamma_2}(0,\delta;W^{2,\rho_2})} \le 2\|u_0\|_{H^2}.$$

Proposition 2.2 (critical case) *Assume that $n > 4$, $a \in L^\infty(0,\infty)$, $\alpha = \frac{8}{n-4}$, $2 \le \beta < \frac{8}{n-4}$, $(\gamma^*,\rho^*) = (\frac{2n}{n-4}, \frac{2n^2}{n^2-4n+16})$, $(\gamma_2,\rho_2) = (\frac{8(\beta+2)}{n\beta}, \beta+2)$. For any $u_0 \in H^2(R^n)$, there exists δ such that the Cauchy problem* (1.1) *has a unique solution u_ε in the space $L^\infty(0,\delta;H^2(R^n)) \cap L^{\gamma^*}(0,\delta;W^{2,\rho^*}(R^n)) \cap L^{\gamma_2}(0,\delta;W^{2,\rho_2}(R^n))$. Moreover,*

$$\|u_\varepsilon\|_{L^\infty(0,\delta;H^2)\cap L^{\gamma^*}(0,\delta;W^{2,\rho^*})\cap L^{\gamma_2}(0,\delta;W^{2,\rho_2})} \le 3\big\|U(t)u_0\big\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})}.$$

3 Main results

Lemma 3.1 *Let n, α, β, (γ_1,ρ_1), (γ_2,ρ_2) be as in Proposition* 2.1. *Assume that u is the solution of* (1.2), *defined on a maximal time interval $[0,T^*)$, $0 < l < T^*$, and u_ε exists on $[0,l]$. If $\limsup_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(0,l;H^2)\cap L^{\gamma_1}(0,l;W^{2,\rho_1})\cap L^{\gamma_2}(0,l;W^{2,\rho_2})} < +\infty$, then we have $u_\varepsilon \to u$ in $L^q(0,l;W^{2,r}(R^n))$ as $\varepsilon \to 0$, where (q,r) is arbitrary admissible pair.*

Proof First, we prove

$$\|u_\varepsilon - u\|_{L^q(0,l;L^r)} \to 0 \quad \text{as } \varepsilon \to 0.$$

From (2.1) and (2.2), using Strichartz estimates, we have

$$\begin{aligned}
&\|u_\varepsilon - u\|_{L^q(0,l;L^r)} \\
&\quad \le \big\|J(t)\big\|_{L^q(0,l;L^r)} + \big\|K(t)\big\|_{L^q(0,l;L^r)} \\
&\quad \le C\big\||u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u\big\|_{L^{\gamma'}(0,l;L^{\rho'})} + C\varepsilon\|a\|_{L^\infty(0,l)}\big\||u_\varepsilon|^\beta u_\varepsilon\big\|_{L^{\gamma_2'}(0,l;L^{\rho_2'})},
\end{aligned} \tag{3.1}$$

where $J(t) = i\lambda\int_0^t U(t-s)(|u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u)(s)\,ds$, $K(t) = -\varepsilon\int_0^t U(t-s)a(s)(|u_\varepsilon|^\beta u_\varepsilon)(s)\,ds$, $(\gamma,\rho) = (\frac{8(\alpha+2)}{n\alpha}, \alpha+2)$.

Since $\limsup_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(0,l;H^2)\cap L^{\gamma_1}(0,l;W^{2,\rho_1})\cap L^{\gamma_2}(0,l;W^{2,\rho_2})} < +\infty$, there exist N_1, ε_0 such that

$$\|u_\varepsilon\|_{L^\infty(0,l;H^2)\cap L^{\gamma_1}(0,l;W^{2,\rho_1})\cap L^{\gamma_2}(0,l;W^{2,\rho_2})} \le N_1 \quad \text{for all } \varepsilon < \varepsilon_0.$$

Let $N_2 = \|u\|_{L^\infty(0,l;H^2)}$, it is obvious that $N_2 < +\infty$. Using the Hölder inequality and the Sobolev embedding [12], we have

$$\begin{aligned}
\big\||u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u\big\|_{L^{\gamma'}(0,l;L^{\rho'})} &\le C\big(\|u_\varepsilon\|^\alpha_{L^a(0,l;L^{\alpha+2})} + \|u\|^\alpha_{L^a(0,l;L^{\alpha+2})}\big)\|u_\varepsilon - u\|_{L^\gamma(0,l;L^\rho)} \\
&\le C\big(\|u_\varepsilon\|^\alpha_{L^\infty(0,l;H^2)} + \|u\|^\alpha_{L^\infty(0,l;H^2)}\big)\|u_\varepsilon - u\|_{L^\gamma(0,l;L^\rho)} \\
&\le C\big(N_1^\alpha + N_2^\alpha\big)\|u_\varepsilon - u\|_{L^\gamma(0,l;L^\rho)},
\end{aligned} \tag{3.2}$$

where $a = \frac{4\alpha(\alpha+2)}{8-(n-4)\alpha}$.

Similarly, we have

$$\big\| |u_\varepsilon|^\beta u_\varepsilon \big\|_{L^{\gamma_2'}(0,l;L^{\rho_2'})} \le \|u_\varepsilon\|^\beta_{L^b(0,l;L^{\beta+2})} \|u_\varepsilon\|_{L^{\gamma_2}(0,l;L^{\rho_2})} \le \|u_\varepsilon\|^\beta_{L^\infty(0,l;H^2)} \|u_\varepsilon\|_{L^{\gamma_2}(0,l;L^{\rho_2})} \le N_1^{\beta+1}, \tag{3.3}$$

where $b = \frac{4\beta(\beta+2)}{8-(n-4)\beta}$.

Let $N_3 = C\|a\|_{L^\infty} N_1^{\beta+1}$. Substituting (3.2) and (3.3) into (3.1), we have

$$\|u_\varepsilon - u\|_{L^q(0,l;L^r)} \le C\big(N_1^\alpha + N_2^\alpha\big) \|u_\varepsilon - u\|_{L^\gamma(0,l;L^\rho)} + \varepsilon N_3. \tag{3.4}$$

In the following we will prove that $\|u_\varepsilon - u\|_{L^\gamma(0,l;L^\rho)} \to 0$ as $\varepsilon \to 0$.

Noting that $N_1, N_2 < \infty$, we can divide the time interval $[0,l]$ into subintervals $[t_i, t_{i+1}]$, $i = 0,1,\dots,J-1$, where $t_0 = 0$, $t_{J-1} = l$ such that in each part $C(\|u_\varepsilon\|^\alpha_{L^a(t_i,t_{i+1};L^{\alpha+2})} + \|u\|^\alpha_{L^a(t_i,t_{i+1};L^{\alpha+2})}) = \frac{1}{2}$.

On $[t_0, t_1]$, since $u_\varepsilon(t_0) = u(t_0) = u_0$, we have

$$\|u_\varepsilon - u\|_{L^\gamma(t_0,t_1;L^\rho)} \le \frac{1}{2} \|u_\varepsilon - u\|_{L^\gamma(t_0,t_1;L^\rho)} + \varepsilon N_3,$$

which means

$$\|u_\varepsilon - u\|_{L^\gamma(t_0,t_1;L^\rho)} \le 2\varepsilon N_3.$$

By (3.4), we have

$$\|u_\varepsilon - u\|_{L^\infty(t_0,t_1;L^2)} \le 2\varepsilon N_3.$$

On $[t_1, t_2]$, we have

$$\begin{aligned} \|u_\varepsilon - u\|_{L^\gamma(t_1,t_2;L^\rho)} &\le \big\|u_\varepsilon(t_1) - u(t_1)\big\|_{L^2} + \frac{1}{2}\|u_\varepsilon - u\|_{L^\gamma(t_1,t_2;L^\rho)} + \varepsilon N_3 \\ &\le 3\varepsilon N_3 + \frac{1}{2}\|u_\varepsilon - u\|_{L^\gamma(t_1,t_2;L^\rho)}, \end{aligned}$$

from which we can obtain

$$\|u_\varepsilon - u\|_{L^\gamma(t_1,t_2;L^\rho)} \le 6\varepsilon N_3.$$

Especially, we have $\|u_\varepsilon - u\|_{L^\infty(t_1,t_2;L^2)} \le 6\varepsilon N_3$.

By induction, we have

$$\begin{aligned} &\|u_\varepsilon - u\|_{L^\gamma(t_i,t_{i+1};L^\rho)} \le 2\big(2^{i+1} - 1\big)\varepsilon N_3, \\ &\|u_\varepsilon - u\|_{L^\infty(t_i,t_{i+1};L^2)} \le 2\big(2^{i+1} - 1\big)\varepsilon N_3, \quad \text{for } i = 0,1,\dots,J-1. \end{aligned}$$

So we have

$$\|u_\varepsilon - u\|_{L^\gamma(0,l;L^\rho)} \le \sum_{i=0}^{J-1} 2\big(2^{i+1} - 1\big)\varepsilon N_3 = \big[4\big(2^J - 1\big) - 2J\big]\varepsilon N_3 \to 0.$$

Furthermore, we have

$$\|u_\varepsilon - u\|_{L^q(0,l;L^r)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Second, we prove

$$\|\nabla u_\varepsilon - \nabla u\|_{L^q(0,l;L^r)} \to 0 \quad \text{as } \varepsilon \to 0.$$

From (2.1) and (2.2), we have

$$\nabla(u_\varepsilon - u) = i\lambda \int_0^t U(t-s)\nabla\bigl(|u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u\bigr)(s)\,ds - \varepsilon \int_0^t U(t-s)a(s)\nabla\bigl(|u_\varepsilon|^\beta u_\varepsilon\bigr)(s)\,ds.$$

Let $g_1(u) = |u|^\alpha u$, $g_2(u) = |u|^\beta u$. Then, using Strichartz estimates, we have

$$\begin{aligned}
&\bigl\|\nabla(u_\varepsilon - u)\bigr\|_{L^q(0,l;L^r)} \\
&\quad\le C\left\|\int_0^t U(t-s)\nabla\bigl(g_1(u_\varepsilon) - g_1(u)\bigr)(s)\,ds\right\|_{L^q(0,l;L^r)} \\
&\qquad + C\varepsilon\left\|\int_0^t U(t-s)a(s)\nabla g_2(u_\varepsilon)(s)\,ds\right\|_{L^q(0,l;L^r)} \\
&\quad\le C\bigl\|\nabla\bigl(g_1(u_\varepsilon) - g_1(u)\bigr)\bigr\|_{L^{\gamma_1'}(0,l;L^{\rho_1'})} + C\varepsilon\|a\|_{L^\infty(0,l)}\bigl\|\nabla g_2(u_\varepsilon)\bigr\|_{L^{\gamma_2'}(0,l;L^{\rho_2'})} \\
&\quad\le C\bigl\|g_1'(u_\varepsilon)\nabla(u_\varepsilon - u)\bigr\|_{L^{\gamma_1'}(0,l;L^{\rho_1'})} + C\bigl\|\bigl(g_1'(u_\varepsilon) - g_1'(u)\bigr)\nabla u\bigr\|_{L^{\gamma_1'}(0,l;L^{\rho_1'})} \\
&\qquad + C\varepsilon\|a\|_{L^\infty(0,l)}\bigl\|\nabla g_2(u_\varepsilon)\bigr\|_{L^{\gamma_2'}(0,l;L^{\rho_2'})}.
\end{aligned} \tag{3.5}$$

Using the Hölder inequality, the Sobolev embedding, and the Young inequality, we obtain

$$\begin{aligned}
&\bigl\|g_1'(u_\varepsilon)\nabla(u_\varepsilon - u)\bigr\|_{L^{\gamma_1'}(0,l;L^{\rho_1'})} \\
&\quad\le C\|u_\varepsilon\|^\alpha_{L^{\gamma_1}(0,l;L^c)}\bigl\|\nabla(u_\varepsilon - u)\bigr\|_{L^{\gamma_1}(0,l;L^{\rho_1})} \\
&\quad\le C\|u_\varepsilon\|^\alpha_{L^{\gamma_1}(0,l;W^{2,\rho_1})}\bigl\|\nabla(u_\varepsilon - u)\bigr\|_{L^{\gamma_1}(0,l;L^{\rho_1})} \quad \left(c = \frac{\rho_1\alpha}{\rho_1 - 2}\right),
\end{aligned} \tag{3.6}$$

$$\bigl\|\nabla g_2(u_\varepsilon)\bigr\|_{L^{\gamma_2'}(0,l;L^{\rho_2'})} \le C\|u_\varepsilon\|^\beta_{L^\infty(0,l;H^2)}\|u_\varepsilon\|_{L^{\gamma_2}(0,l;W^{2,\rho_2})}, \tag{3.7}$$

and

$$\begin{aligned}
&\bigl\|\bigl(g_1'(u_\varepsilon) - g_1'(u)\bigr)\nabla u\bigr\|_{L^{\gamma_1'}(0,l;L^{\rho_1'})} \\
&\quad\le C\bigl(\|u_\varepsilon\|^{\alpha-1}_{L^{\gamma_1}(0,l;L^{d_1})} + \|u\|^{\alpha-1}_{L^{\gamma_1}(0,l;L^{d_1})}\bigr)\|u_\varepsilon - u\|_{L^{\gamma_1}(0,l;L^{e_1})}\|\nabla u\|_{L^{\gamma_1}(0,l;L^{e_1})} \\
&\quad\le C\bigl(\|u_\varepsilon\|^{\alpha-1}_{L^{\gamma_1}(0,l;W^{2,\rho_1})} + \|u\|^{\alpha-1}_{L^{\gamma_1}(0,l;W^{2,\rho_1})}\bigr)\bigl\|\nabla(u_\varepsilon - u)\bigr\|_{L^{\gamma_1}(0,l;L^{\rho_1})}\|u\|_{L^{\gamma_1}(0,l;W^{2,\rho_1})} \\
&\quad\le C\bigl(\|u_\varepsilon\|^{\alpha}_{L^{\gamma_1}(0,l;W^{2,\rho_1})} + \|u\|^{\alpha}_{L^{\gamma_1}(0,l;W^{2,\rho_1})}\bigr)\bigl\|\nabla(u_\varepsilon - u)\bigr\|_{L^{\gamma_1}(0,l;L^{\rho_1})},
\end{aligned} \tag{3.8}$$

where $d_1 = \frac{2n(\alpha+2)(\alpha-1)}{24-(n-4)(\alpha+2)}$, $e_1 = \frac{2n(\alpha+2)}{(n-2)(\alpha+2)-8}$.

Substituting (3.6)-(3.8) into (3.5), we have

$$\left\|\nabla(u_\varepsilon - u)\right\|_{L^q(0,l;L^r)} \leq C\big(\|u_\varepsilon\|^\alpha_{L^{\gamma_1}(0,l;W^{2,\rho_1})} + \|u\|^\alpha_{L^{\gamma_1}(0,l;W^{2,\rho_1})}\big)\left\|\nabla(u_\varepsilon - u)\right\|_{L^{\gamma_1}(0,l;L^{\rho_1})} + \varepsilon N_3.$$

Similar to the proof in the first step, we have

$$\left\|\nabla(u_\varepsilon - u)\right\|_{L^q(0,l;L^r)} \to 0 \quad \text{as } \varepsilon \to 0.$$

At last, we prove

$$\left\|D^2 u_\varepsilon - D^2 u\right\|_{L^q(0,l;L^r)} \to 0 \quad \text{as } \varepsilon \to 0.$$

By simple computing, we have

$$D^2 u_\varepsilon - D^2 u = i(K_1 + K_2 + K_3), \tag{3.9}$$

where $K_1 = \lambda \int_0^t U(t-s)A_1(u_\varepsilon, u)(s)\,ds$, $K_2 = \lambda \int_0^t U(t-s)A_2(u_\varepsilon, u)(s)\,ds$, $K_3 = -\varepsilon \int_0^t U(t-s)a(s)A_3(u_\varepsilon)(s)\,ds$. The arrays $A_1(u_\varepsilon, u) = g_1'(u_\varepsilon)D^2(u_\varepsilon - u) + g_1''(u_\varepsilon)D(u_\varepsilon - u) \times Du$, $A_2(u_\varepsilon, u) = Du \times [g_1''(u_\varepsilon)Du_\varepsilon - g_1''(u)Du] + [g_1'(u_\varepsilon) - g_1'(u)]D^2 u$, $A_3(u_\varepsilon) = g_2''(u_\varepsilon)Du_\varepsilon \times Du_\varepsilon + g_2'(u_\varepsilon)D^2 u_\varepsilon$.

By the Hölder inequality and the Sobolev embedding, we have

$$\left\|g_1'(u_\varepsilon)D^2(u_\varepsilon - u)\right\|_{L^{\gamma_1'}(0,l;L^{\rho_1'})} \leq \|u_\varepsilon\|^\alpha_{L^{\gamma_1}(0,l;W^{2,\rho_1})}\left\|D^2(u_\varepsilon - u)\right\|_{L^{\gamma_1}(0,l;L^{\rho_1})} \tag{3.10}$$

and

$$\begin{aligned}
&\left\|g_1''(u_\varepsilon)D(u_\varepsilon - u)^\perp \times Du_\varepsilon\right\|_{L^{\gamma_1'}(0,l;L^{\rho_1'})} \\
&\quad \leq \|u_\varepsilon\|^{\alpha-1}_{L^{\gamma_1}(0,l;L^{d_1})}\left\|D(u_\varepsilon - u)\right\|_{L^{\gamma_1}(0,l;L^{e_1})}\|Du_\varepsilon\|_{L^{\gamma_1}(0,l;L^{e_1})} \\
&\quad \leq \|u_\varepsilon\|^\alpha_{L^{\gamma_1}(0,l;W^{2,\rho_1})}\left\|D^2(u_\varepsilon - u)\right\|_{L^{\gamma_1}(0,l;L^{\rho_1})}.
\end{aligned} \tag{3.11}$$

Thus we have from (3.10) and (3.11)

$$\begin{aligned}
\|K_1\|_{L^q(0,l;L^r)} &\leq \left\|g_1'(u_\varepsilon)D^2(u_\varepsilon - u)\right\|_{L^{\gamma_1'}(0,l;L^{\rho_1'})} + \left\|g_1''(u_\varepsilon)D(u_\varepsilon - u)^\perp \times Du_\varepsilon\right\|_{L^{\gamma_1'}(0,l;L^{\rho_1'})} \\
&\leq C\|u_\varepsilon\|^\alpha_{L^{\gamma_1}(0,l;W^{2,\rho_1})}\left\|D^2(u_\varepsilon - u)\right\|_{L^{\gamma_1}(0,l;L^{\rho_1})}.
\end{aligned} \tag{3.12}$$

Similar to the proof of (3.11), we obtain

$$\begin{aligned}
&\left\|g_1''(u_\varepsilon)D(u_\varepsilon - u)^\perp \times Du\right\|_{L^{\gamma_1'}(0,l;L^{\rho_1'})} \\
&\quad \leq \big(\|u\|^\alpha_{L^{\gamma_1}(0,l;W^{2,\rho_1})} + \|u_\varepsilon\|^\alpha_{L^{\gamma_1}(0,l;W^{2,\rho_1})}\big)\left\|D^2(u_\varepsilon - u)\right\|_{L^{\gamma_1}(0,l;L^{\rho_1})}.
\end{aligned} \tag{3.13}$$

Noting that $\alpha \geq 2$, we have

$$\begin{aligned}
&\left\|\big(g_1''(u_\varepsilon) - g_1''(u)\big)Du^\perp \times Du\right\|_{L^{\gamma_1'}(0,l;L^{\rho_1'})} \\
&\quad \leq C\left\|\big(|u_\varepsilon|^{\alpha-2} + |u|^{\alpha-2}\big)(u_\varepsilon - u)Du^\perp \times Du\right\|_{L^{\gamma_1'}(0,l;L^{\rho_1'})}
\end{aligned}$$

$$\begin{aligned}
&\le C\big(\|u_\varepsilon\|^{\alpha-2}_{L^{\gamma_1}(0,l;L^c)} + \|u\|^{\alpha-2}_{L^{\gamma_1}(0,l;L^c)}\big)\|u_\varepsilon - u\|_{L^{\gamma_1}(0,l;L^{d_2})}\|Du\|^2_{L^{\gamma_1}(0,l;L^{e_2})} \\
&\le \big(\|u\|^{\alpha}_{L^{\gamma_1}(0,l;W^{2,\rho_1})} + \|u_\varepsilon\|^{\alpha}_{L^{\gamma_1}(0,l;W^{2,\rho_1})}\big)\big\|D^2(u_\varepsilon - u)\big\|_{L^{\gamma_1}(0,l;L^{\rho_1})},
\end{aligned} \tag{3.14}$$

where $e_2 = \frac{2n(\alpha+2)}{(n-2)(\alpha+2)-8}$, $\frac{1}{\rho_1{}'} = \frac{(\rho_1-2)(\alpha-2)}{\rho_1\alpha} + \frac{1}{d_2} + \frac{2}{e_2}$.

Similarly, using the Hölder inequality and the Sobolev embedding, we obtain

$$\begin{aligned}
&\big\|\big(g_1'(u_\varepsilon) - g_1'(u)\big)D^2u\big\|_{L^{\gamma_1{}'}(0,l;L^{\rho_1{}'})} \\
&\quad\le C\big(\|u_\varepsilon\|^{\alpha-1}_{L^{\gamma_1}(0,l;L^c)} + \|u\|^{\alpha-1}_{L^{\gamma_1}(0,l;L^c)}\big)\|u_\varepsilon - u\|_{L^{\gamma_1}(0,l;L^c)}\big\|D^2u\big\|_{L^{\gamma_1}(0,l;L^{\rho_1})} \\
&\quad\le C\big(\|u_\varepsilon\|^{\alpha}_{L^{\gamma_1}(0,l;W^{2,\rho_1})} + \|u\|^{\alpha}_{L^{\gamma_1}(0,l;W^{2,\rho_1})}\big)\big\|D^2(u_\varepsilon - u)\big\|_{L^{\gamma_1}(0,l;L^{\rho_1})}.
\end{aligned} \tag{3.15}$$

Thus we have from (3.13) and (3.15)

$$\begin{aligned}
\|K_2\|_{L^q(0,l;L^r)} &\le \big\|A_2(u_\varepsilon, u)\big\|_{L^{\gamma_1{}'}(0,l;L^{\rho_1{}'})} \\
&\le \big\|g_1''(u_\varepsilon)D(u_\varepsilon - u)^\perp \times Du\big\|_{L^{\gamma_1{}'}(0,l;L^{\rho_1{}'})} \\
&\quad + \big\|\big(g_1''(u_\varepsilon) - g_1''(u)\big)Du^\perp \times Du\big\|_{L^{\gamma_1{}'}(0,l;L^{\rho_1{}'})} \\
&\quad + \big\|\big(g_1'(u_\varepsilon) - g_1'(u)\big)D^2u\big\|_{L^{\gamma_1{}'}(0,l;L^{\rho_1{}'})} \\
&\le C\big(\|u_\varepsilon\|^{\alpha}_{L^{\gamma_1}(0,l;W^{2,\rho_1})} + \|u\|^{\alpha}_{L^{\gamma_1}(0,l;W^{2,\rho_1})}\big)\big\|D^2(u_\varepsilon - u)\big\|_{L^{\gamma_1}(0,l;L^{\rho_1})}.
\end{aligned} \tag{3.16}$$

Similar to the proof of (3.3), we obtain

$$\begin{aligned}
\big\|g_2'(u_\varepsilon)D^2u_\varepsilon\big\|_{L^{\gamma_2'}(0,l;L^{\rho_2'})} &\le \|u_\varepsilon\|^{\beta}_{L^b(0,l;L^{\beta+2})}\big\|D^2u_\varepsilon\big\|_{L^{\gamma_2}(0,l;L^{\rho_2})} \le N_1^{\beta+1} \\
&\le \|u_\varepsilon\|^{\beta}_{L^\infty(0,l;H^2)}\|u_\varepsilon\|_{L^{\gamma_2}(0,l;W^{2,\rho_2})} \le N_1^{\beta+1}
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
\big\|g_2''(u_\varepsilon)Du_\varepsilon^\perp \times Du_\varepsilon\big\|_{L^{\gamma_2'}(0,l;L^{\rho_2'})} &\le \|u_\varepsilon\|^{\beta-1}_{L^b(0,l;L^{\beta+2})}\|Du_\varepsilon\|^2_{L^{\gamma_2}(0,l;L^{\rho^2})} \\
&\le \|u_\varepsilon\|^{\beta+1}_{L^{\gamma_2}(0,l;W^{2,\rho_2})} \le N_1^{\beta+1}.
\end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we immediately obtain

$$\begin{aligned}
\|K_3\|_{L^q(0,l;L^r)} &\le \varepsilon\|a\|_{L^\infty(0,l)}\big\|A_3(u_\varepsilon)\big\|_{L^{\gamma_2'}(0,l;L^{\rho_2'})} \\
&\le \varepsilon\|a\|_{L^\infty(0,l)}\big[\big\|g_2'(u_\varepsilon)D^2u_\varepsilon\big\|_{L^{\gamma_2'}(0,l;L^{\rho_2'})} + \big\|g_2''(u_\varepsilon)Du_\varepsilon^\perp \times Du_\varepsilon\big\|_{L^{\gamma_2'}(0,l;L^{\rho_2'})}\big] \\
&\le \varepsilon N_3.
\end{aligned} \tag{3.19}$$

Taking, respectively, $(q,r) = (\gamma,\rho)$ and $(q,r) = (\gamma_1,\rho_1)$ in (3.9), (3.12), (3.16), and (3.19), similar to the method of the first step, we can obtain

$$\big\|D^2u_\varepsilon - D^2u\big\|_{L^q(0,l;L^r)} \to 0 \quad \text{as } \varepsilon \to 0. \qquad \square$$

Noting that if $\alpha = \frac{8}{n-4}$, a in (3.2) will be meaningless. So we will need the following lemma for the critical case.

Lemma 3.2 *Let n, α, β, (γ^*, ρ^*), (γ_2, ρ_2) be as in Proposition* 2.2. *Assume that u is the solution of* (1.2), *defined on a maximal time interval $[0, T^*)$, $0 < l < T^*$, and u_ε exists on $[0, l]$. If $\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(0,l;H^2)\cap L^{\gamma^*}(0,l;W^{2,\rho^*})\cap L^{\gamma_2}(0,l;W^{2,\rho_2})} < +\infty$, then we have $u_\varepsilon \to u$ in $L^q(0, l; W^{2,r}(R^n))$ as $\varepsilon \to 0$, where (q, r) is arbitrary admissible pair.*

Proof Using the Hölder inequality and a Sobolev embedding, we have

$$\begin{aligned}
&\big\||u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u\big\|_{L^{\gamma^{*\prime}}(0,l;L^{\rho^{*\prime}})} \\
&\quad\le C\big(\|u_\varepsilon\|^\alpha_{L^{\gamma^*}(0,l;L^{\frac{\rho^*\alpha}{\rho^*-2}})} + \|u\|^\alpha_{L^{\gamma^*}(0,l;L^{\frac{\rho^*\alpha}{\rho^*-2}})}\big)\|u_\varepsilon - u\|_{L^{\gamma^*}(0,l;L^{\rho^*})} \\
&\quad\le C\big(\|u_\varepsilon\|^\alpha_{L^{\gamma^*}(0,l;W^{2,\rho^*})} + \|u\|^\alpha_{L^{\gamma^*}(0,l;W^{2,\rho^*})}\big)\|u_\varepsilon - u\|_{L^{\gamma^*}(0,l;L^{\rho^*})}.
\end{aligned} \tag{3.20}$$

From (2.1) and (2.2), using Strichartz estimates, we have

$$\begin{aligned}
\|u_\varepsilon - u\|_{L^q(0,l;L^r)} &\le C\big\||u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u\big\|_{L^{\gamma^{*\prime}}(0,l;L^{\rho^{*\prime}})} + C\varepsilon\|a\|_{L^\infty(0,l)}\big\||u_\varepsilon|^\beta u_\varepsilon\big\|_{L^{\gamma_2'}(0,l;L^{\rho_2'})} \\
&\le C\big(\|u_\varepsilon\|^\alpha_{L^{\gamma^*}(0,l;W^{2,\rho^*})} + \|u\|^\alpha_{L^{\gamma^*}(0,l;W^{2,\rho^*})}\big)\|u_\varepsilon - u\|_{L^{\gamma^*}(0,l;L^{\rho^*})} \\
&\quad + C\varepsilon\|a\|_{L^\infty(0,l)}\big\||u_\varepsilon|^\beta u_\varepsilon\big\|_{L^{\gamma_2'}(0,l;L^{\rho_2'})};
\end{aligned} \tag{3.21}$$

similarly as in Lemma 3.1, we can obtain

$$\|u_\varepsilon - u\|_{L^q(0,l;L^r)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Noting that for (γ_1, ρ_1) in Lemma 3.1 in the case $\alpha = \frac{8}{n-4}$, $2 \le \beta < \frac{8}{n-4}$, we have

$$(\gamma_1, \rho_1) = \big(\gamma^*, \rho^*\big),$$

thus obviously

$$\big\|\nabla(u_\varepsilon - u)\big\|_{L^q(0,l;L^r)} \to 0 \quad \text{as } \varepsilon \to 0$$

and

$$\big\|D^2 u_\varepsilon - D^2 u\big\|_{L^q(0,l;L^r)} \to 0 \quad \text{as } \varepsilon \to 0,$$

for all admissible pairs (q, r). □

Remark 3.1 For the critical case $2 \le \alpha < \frac{8}{n-4}$, $\beta = \frac{8}{n-4}$, we only take the working space as $L^\infty(0, \delta; H^2(R^n)) \cap L^{\gamma_1}(0, \delta; W^{2,\rho_1}(R^n)) \cap L^{\gamma^*}(0, \delta; W^{2,\rho^*}(R^n))$.

For the case $\alpha = \beta = \frac{8}{n-4}$, we take the working space as $L^\infty(0, \delta; H^2(R^n)) \cap L^{\gamma^*}(0, \delta; W^{2,\rho^*}(R^n))$.

Theorem 3.1 *Assume that $n > 4$, $a \in L^\infty(0, \infty)$, $2 \le \alpha \le \frac{8}{n-4}$, and $2 \le \beta \le \frac{8}{n-4}$. Assume that u is the solution of* (1.2) *with initial value $u_0 \in H^2(R^n)$, defined on a maximal time interval $[0, T^*)$. Then we have*:

(1) *For any given* $0 < T < T^*$, *there is a solution* u_ε *on* $[0, T]$.
(2) $u_\varepsilon \to u$ *in* $L^q(0, T; W^{2,r}(R^n))$ *as* $\varepsilon \to 0$, *where* (q, r) *is an arbitrary admissible pair.*

Proof (1) The case $2 \le \alpha < \frac{8}{n-4}$ and $2 \le \beta < \frac{8}{n-4}$.

From Proposition 2.1, we find that there exists u_ε on $[0, \delta]$ such that

$$\|u_\varepsilon\|_{L^\infty(0,\delta;H^2)\cap L^{\gamma_1}(0,\delta;W^{2,\rho_1})\cap L^{\gamma_2}(0,\delta;W^{2,\rho_2})} \le 2\|u_0\|_{H^2}.$$

So for small ε, we have

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(0,\delta;H^2)\cap L^{\gamma_1}(0,\delta;W^{2,\rho_1})\cap L^{\gamma_2}(0,\delta;W^{2,\rho_2})} < +\infty.$$

Using Lemma 3.1, we have $u_\varepsilon \to u$ in $L^q(0, \delta; W^{2,r}(R^n))$ as $\varepsilon \to 0$, for any arbitrary admissible pair (q, r).

Especially, we have $\|u_\varepsilon(\delta)\|_{H^2} \le 2\|u_0\|_{H^2}$. Again using Proposition 2.1, there exists u_ε on $[\delta, 2\delta]$ such that

$$\|u_\varepsilon\|_{L^\infty(\delta,2\delta;H^2)\cap L^{\gamma_1}(\delta,2\delta;W^{2,\rho_1})\cap L^{\gamma_2}(\delta,2\delta;W^{2,\rho_2})} \le 2\left\|u_\varepsilon(\delta)\right\|_{H^2} \le 2\|u_0\|_{H^2}.$$

By a continuation extension method, we obtain the solution u_ε on $[0, T]$ $(0 < T < T^*)$ such that

$$\|u_\varepsilon\|_{L^\infty(0,T;H^2)\cap L^{\gamma}(0,T;W^{2,\rho})\cap L^{\gamma_1}(0,T;W^{2,\rho_1})} \le 2\|u_0\|_{H^2}.$$

So

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(0,T;H^2)\cap L^{\gamma_1}(0,T;W^{2,\rho_1})\cap L^{\gamma_2}(0,T;W^{2,\rho_2})} < +\infty,$$

using Lemma 3.1, we immediately have $u_\varepsilon \to u$ in $L^q(0, T; W^{2,r}(R^n))$ as $\varepsilon \to 0$, for any arbitrary admissible pair (q, r).

(2) *Case* 1: $\alpha = \frac{8}{n-4}$, $2 \le \beta < \frac{8}{n-4}$.

From Proposition 2.2, we find that there exists u_ε on $[0, \delta]$ such that

$$\|u_\varepsilon\|_{L^\infty(0,\delta;H^2)\cap L^{\gamma^*}(0,\delta;W^{2,\rho^*})\cap L^{\gamma_2}(0,\delta;W^{2,\rho_2})} \le 3\left\|U(t)u_0\right\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})}.$$

So for small ε, we have

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(0,\delta;H^2)\cap L^{\gamma^*}(0,\delta;W^{2,\rho^*})\cap L^{\gamma_2}(0,\delta;W^{2,\rho_2})} < +\infty.$$

Using Lemma 3.2, we have $u_\varepsilon \to u$ in $L^q(0, \delta; W^{2,r}(R^n))$ as $\varepsilon \to 0$, for any arbitrary admissible pair (q, r).

Noting that

$$\left\|U(t)u_\varepsilon(\delta)\right\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})} \le C\left\|u_\varepsilon(\delta)\right\|_{H^2} \le 3C\left\|U(t)u_0\right\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})},$$

so, again using Proposition 2.2, there exists u_ε on $[\delta, 2\delta]$ such that

$$\begin{aligned}\|u_\varepsilon\|_{L^\infty(\delta,2\delta;H^2)\cap L^{\gamma^*}(\delta,2\delta;W^{2,\rho^*})\cap L^{\gamma_2}(\delta,2\delta;W^{2,\rho_2})} &\leq 3C\big\|U(t)u_\varepsilon(\delta)\big\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})}\\ &\leq (3C)^2\big\|U(t)u_0\big\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})}.\end{aligned}$$

By continuation extension method, we obtain the solution u_ε on $[0, T]$ $(0 < T < T^*)$ such that

$$\|u_\varepsilon\|_{L^\infty(0,T;H^2)\cap L^{\gamma^*}(0,T;W^{2,\rho^*})\cap L^{\gamma_2}(0,T;W^{2,\rho_2})} \leq C(T)\big\|U(t)u_0\big\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})}.$$

So

$$\limsup_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(0,T;H^2)\cap L^{\gamma^*}(0,T;W^{2,\rho^*})\cap L^{\gamma_2}(0,T;W^{2,\rho_2})} \leq C(T)\big\|U(t)u_0\big\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})} < +\infty,$$

using Lemma 3.2, we immediately have $u_\varepsilon \to u$ in $L^q(0, T; W^{2,r}(R^n))$ as $\varepsilon \to 0$, for any arbitrary admissible pair (q, r).

Case 2: $\beta = \frac{8}{n-4}$, $2 \leq \alpha < \frac{8}{n-4}$ or $\alpha = \beta = \frac{8}{n-4}$.

See Remark 3.1, the proof is similar; here we omit it. □

Lemma 3.3 *Assume that u is the global solution of* (1.2) *with the initial valve $u_0 \in H^2(R^n)$ and $u \in L^q_{loc}(0,\infty; W^{2,r}(R^n))$. Then we have*:

(1) *The solution u_ε of* (1.1) *with the initial valve u_0 is global for sufficiently small ε.*

(2) *$u_\varepsilon \to u$ in $L^q(0,\infty; W^{2,r}(R^n))$ as $\varepsilon \to 0$, where (q, r) is an arbitrary admissible pair.*

Proof (1) We will prove that u_ε is also global for small ε if u is global.

From Theorem 3.1, we can see

$$\big\|u_\varepsilon(T) - u(T)\big\|_{H^2} \to 0 \quad \text{as } \varepsilon \to 0$$

for all $T < \infty$.

Since u is global, for any $\eta > 0$, there exists sufficient large T such that

$$\|u\|_{L^{\gamma_1}(T,\infty;W^{2,\rho_1})} \leq \frac{\eta}{4},$$

(γ_1, ρ_1) is the same as in Theorem 3.1.

Case 1: $2 \leq \alpha < \frac{8}{n-4}$, $2 \leq \beta \leq \frac{8}{n-4}$.

From (2.2), (2.3)-(2.4), using a continuity argument we can obtain

$$\begin{aligned}&\big\|U(t)u(T)\big\|_{L^{\gamma_1}(0,\infty;W^{2,\rho_1})}\\ &\quad\leq C\big\|u(T)\big\|_{L^{\gamma_1}(0,\infty;W^{2,\rho_1})} + C\bigg\|\int_T^t U(t-\tau)|u|^\alpha u(\tau)\,d\tau\bigg\|_{L^{\gamma_1}(0,\infty;W^{2,\rho_1})}\\ &\quad\leq C\big\|u(t)\big\|_{L^{\gamma_1}(T,\infty;W^{2,\rho_1})} + C\big\||u|^{\alpha+1}\big\|_{L^{\gamma_1}(T,\infty;W^{2,\rho_1})}\\ &\quad\leq \frac{\eta}{2}.\end{aligned}$$

Thus we have

$$\begin{aligned}\|U(t)u_\varepsilon(T)\|_{L^{\gamma_1}(0,\infty;W^{2,\rho_1})} &\le \|U(t)(u_\varepsilon(T)-u(T))\|_{L^{\gamma_1}(0,\infty;W^{2,\rho_1})}\\ &\quad+\|U(t)u(T)\|_{L^{\gamma_1}(0,\infty;W^{2,\rho_1})}\le\eta.\end{aligned}$$

Obviously $\|U(t)u_\varepsilon(T)\|_{L^q(0,\infty;W^{2,r})}\le\eta$ for suitable T and any admissible pair (q,r).

Furthermore we define the working space as follows:

$$\begin{aligned}X(0,t)={}&L^{\frac{2(n+4)}{n}}\bigl(0,t;L^{\frac{2(n+4)}{n}}(R^n)\bigr)\cap L^{\frac{2(n+4)}{n-4}}\bigl(0,t;W^{2,\frac{2n(n+4)}{n^2+16}}(R^n)\bigr)\\ &\cap L^{\frac{2(n+4)}{n-2}}\bigl(0,t;W^{2,\frac{2n(n+4)}{n^2+8}}(R^n)\bigr)\cap L^{\gamma_2}\bigl(0,t;W^{2,\rho_2}(R^n)\bigr)\cap L^\infty\bigl(0,t;H^2(R^n)\bigr),\end{aligned}$$

where (γ_2,ρ_2) is the same as in Theorem 3.1.

Using the Hölder inequality, the interpolation inequality [13], and the Sobolev embedding, we have

$$\begin{aligned}&\bigl\||u_\varepsilon|^\alpha u_\varepsilon\bigr\|_{L^{\frac{2(n+4)}{n+8}}(0,t;L^{\frac{2(n+4)}{n+8}})}\\ &\quad\le\|u_\varepsilon\|^\alpha_{L^{\frac{(n+4)\alpha}{4}}(0,t;L^{\frac{(n+4)\alpha}{4}})}\|u_\varepsilon\|_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})}\\ &\quad\le C\|u_\varepsilon\|^{2-\frac{(n-4)\alpha}{4}}_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})}\|u_\varepsilon\|^{\frac{n\alpha}{4}-2}_{L^{\frac{2(n+4)}{n-4}}(0,t;L^{\frac{2(n+4)}{n-4}})}\|u_\varepsilon\|_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})}\\ &\quad\le C\|u_\varepsilon\|^{3-\frac{(n-4)\alpha}{4}}_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})}\|u_\varepsilon\|^{\frac{n\alpha}{4}-2}_{L^{\frac{2(n+4)}{n-4}}(0,t;W^{2,\frac{2n(n+4)}{n^2+16}})}\\ &\quad\le C\|u_\varepsilon\|^{\alpha+1}_{X(0,t)}.\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}&\bigl\|\nabla\bigl(|u_\varepsilon|^\alpha u_\varepsilon\bigr)\bigr\|_{L^{\frac{2(n+4)}{n+8}}(0,t;L^{\frac{2(n+4)}{n+8}})}\le C\|u_\varepsilon\|^{\alpha+1}_{X(0,t)};\\ &\bigl\||u_\varepsilon|^{\alpha-1}D^2u_\varepsilon\bigr\|_{L^{\frac{2(n+4)}{n+8}}(0,t;L^{\frac{2(n+4)}{n+8}})}\le C\|u_\varepsilon\|^{\alpha+1}_{X(0,t)}.\end{aligned}$$

For the case $4<n<8$, we have

$$\begin{aligned}&\bigl\||u_\varepsilon|^{\alpha-1}\nabla u_\varepsilon\cdot\nabla u_\varepsilon\bigr\|_{L^{\frac{2(n+4)}{n+8}}(0,t;L^{\frac{2(n+4)}{n+8}})}\\ &\quad\le\bigl\||u_\varepsilon|^{\alpha-1}\bigr\|_{L^{\frac{2(n+4)}{8-n}}(0,t;L^{\frac{2(n+4)}{8-n}})}\|\nabla u_\varepsilon\|^2_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})}\\ &\quad\le C\|u_\varepsilon\|^{\frac{8-n}{4}-\frac{(n-4)(\alpha-1)}{4}}_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})}\|u_\varepsilon\|^{\frac{n(\alpha-1)}{4}-\frac{8-n}{4}}_{L^{\frac{2(n+4)}{n-4}}(0,t;W^{2,\frac{2(n+4)}{n^2+16}})}\|u_\varepsilon\|^2_{L^{\frac{2(n+4)}{n}}(0,t;W^{2,\frac{2(n+4)}{n}})}\\ &\quad\le C\|u_\varepsilon\|^{\alpha+1}_{X(0,t)}.\end{aligned}$$

For the case $8\le n<12$, we have

$$\begin{aligned}&\bigl\||u_\varepsilon|^{\alpha-1}\nabla u_\varepsilon\cdot\nabla u_\varepsilon\bigr\|_{L^{\frac{2(n+4)}{n+8}}(0,t;L^{\frac{2(n+4)}{n+8}})}\\ &\quad\le\bigl\||u_\varepsilon|^{\alpha-1}\bigr\|_{L^{\frac{2(n+4)}{12-n}}(0,t;L^{\frac{2(n+4)}{12-n}})}\|\nabla u_\varepsilon\|^2_{L^{\frac{2(n+4)}{n-2}}(0,t;L^{\frac{2(n+4)}{n-2}})}\end{aligned}$$

$$\leq C\|u_\varepsilon\|^{\frac{12-n}{4}-\frac{(n-4)(\alpha-1)}{4}}_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})}\|u_\varepsilon\|^{\frac{n(\alpha-1)}{4}-\frac{12-n}{4}}_{L^{\frac{2(n+4)}{n-2}}(0,t;W^{2,\frac{2(n+4)}{n^2+8}})}\|u_\varepsilon\|^2_{L^{\frac{2(n+4)}{n}}(0,t;W^{2,\frac{2(n+4)}{n}})}$$

$$\leq C\|u_\varepsilon\|^{\alpha+1}_{X(0,t)};$$

thus we have

$$\left\|D^2\left(|u_\varepsilon|^\alpha u_\varepsilon\right)\right\|_{L^{\frac{2(n+4)}{n+8}}(0,t;L^{\frac{2(n+4)}{n+8}})}\leq C\|u_\varepsilon\|^{\alpha+1}_{X(0,t)}.$$

Noting that $(\frac{2(n+4)}{n+8},\frac{2(n+4)}{n+8})$ is an admissible pair, using Strichartz estimates, we can obtain

$$\left\|\int_T^t U(t-\tau)|u_\varepsilon|^\alpha u_\varepsilon\right\|_{X(0,t)}\leq C\left\||u_\varepsilon|^\alpha u_\varepsilon\right\|_{L^{\frac{2(n+4)}{n-8}}(0,t;W^{2,\frac{2(n+4)}{n-8}})}\leq C\|u_\varepsilon\|^{\alpha+1}_{X(0,t)}.$$

Using (2.1), we have

$$\left\|u_\varepsilon(T)\right\|_{X(0,t)}\leq C\left\|U(t)u_\varepsilon(T)\right\|_{X(0,t)}+C\|u_\varepsilon\|^{\alpha+1}_{X(0,t)}+C\|u_\varepsilon\|^{\beta+1}_{X(0,t)}.$$

Using a continuity argument, we immediately have

$$\left\|u_\varepsilon(T)\right\|_{X(0,t)}\leq 3\eta\quad\text{for sufficiently small }\varepsilon,$$

which means that $\|u_\varepsilon\|_{X(T,\infty)}\leq M$, where M is a constant.

Furthermore, we have $\|u_\varepsilon\|_{L^q(T,\infty;W^{2,r})}\leq M$, for any admissible pair (q,r). Thus u_ε is global.

Case 2: $\alpha=\frac{8}{n-4}$, $2\leq\beta\leq\frac{8}{n-4}$.

We need the following working space:

$$Y(0,t)=L^{\frac{2n}{n-4}}\left(0,t;W^{2,\frac{2n^2}{n^2-4n+16}}\left(R^n\right)\right)\cap L^{\gamma_2}\left(0,t;W^{2,\rho_2}\left(R^n\right)\right)\cap L^\infty\left(0,t;H^2\left(R^n\right)\right).$$

The process of proof is similar to the case 1, so here we omit the detailed proof.

(2) In the sequel, we prove $u_\varepsilon\to u$ in $L^q(0,\infty;W^{2,r}(R^n))$ as $\varepsilon\to 0$, for any admissible pair (q,r).

Using (2.1) and (2.2), we have

$$\begin{aligned}
&u_\varepsilon(T+t)-u(T+t)\\
&\quad=U(t)\left(u_\varepsilon(T)-u(T)\right)+i\int_0^t U(t-\tau)\left(|u_\varepsilon|^\alpha u_\varepsilon-|u|^\alpha u\right)(T+\tau)\,d\tau\\
&\qquad+\varepsilon\int_0^t U(t-\tau)a(T+\tau)|u|^\beta u\left((T+\tau)\right)d\tau\\
&\quad=a(t)+b(t)+c(t);
\end{aligned}$$

$$\left\|a(t)\right\|_{L^q(0,\infty;W^{2,r})}\leq C\left\|u_\varepsilon(T)-u(T)\right\|_{H^2}\to 0\quad\text{as }\varepsilon\to 0;$$

$$\begin{aligned}
\left\|b(t)\right\|_{L^q(0,\infty;W^{2,r})}&\leq C\left\||u_\varepsilon|^\alpha u_\varepsilon-|u|^\alpha u\right\|_{L^{\frac{2(n+4)}{n+8}}(0,\infty;W^{2,\frac{2(n+4)}{n+8}})}\\
&\leq C\left(\|u_\varepsilon\|^\alpha_{X(0,\infty)}+\|u\|^\alpha_{X(0,\infty)}\right)\|u_\varepsilon-u\|_{X(0,\infty)}\to 0;
\end{aligned}$$

$$\left\|c(t)\right\|_{L^q(0,\infty;W^{2,r})} \le C\varepsilon\|u_\varepsilon\|_{X(0,\infty)}^{\beta+1}.$$

Thus we have

$$\left\|u_\varepsilon(T+t)-u(T+t)\right\|_{L^q(0,\infty;W^{2,r})} \to 0 \quad \text{as } \varepsilon \to 0. \qquad \square$$

Theorem 3.2 *Assume that* $n > 4$, $a \in L^\infty(0,\infty)$, $2 \le \alpha \le \frac{8}{n-4}$, *and* $2 \le \beta \le \frac{8}{n-4}$. *One of the following conditions holds*:

(i) $\lambda < 0$,

(ii) $\lambda > 0$, $\|u_0\|_{H^2}$ *is small*.

Then we have

(1) *The solution* u_ε *of* (1.1) *is global for small* ε.

(2) $u_\varepsilon \to u$ *in* $L^q(0,\infty;W^{2,r}(R^n))$ *as* $\varepsilon \to 0$, *where* (q,r) *is arbitrary admissible pair*.

Proof Note that the solution u of (1.2) is global provided the conditions (i) $\lambda < 0$ or (ii) $\lambda > 0$, $\|u_0\|_{H^2}$ is small hold. Combing Lemma 3.3, the proof of Theorem 3.2 immediately is complete. $\square$

4 Conclusions

The appearance of gain/loss does not affect the local well-posedness of the solution. Moreover, the solution u_ε will converge to u in the space $L^q(0,T;W^{2,r}(R^n))$ as ε converges to 0. Furthermore, if (i) $\lambda < 0$, or (ii) $\lambda > 0$, $\|u_0\|_{H^2}$ is small, then we have found that the global solution u_ε will converge to u in the space $L^q(0,\infty;W^{2,r}(R^n))$ as ε converges to 0.

Competing interests

The author declares that they have no competing interests.

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